2019-Oct-31 MATB44-H3 Prof. Yun William Yu

Thursday, October 31, 2019 4:50 AM

Outline:

- Review of power series
- Series methods for ODEs

A **power series** is a series of the form

$$a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$$

where a_i's and x_0 are constants and x is a variable.

The **interval of convergence** of a power series is the set $I \subset \mathbb{R}$ for which the series converges.

Recall from calculus that there are several tests you can apply to determine the interval of convergence (e.g. root test, ratio test, integral test)

Ex Ratio test states that a series $u_1 + u_2 + \cdots + u_n$ converges absolutely if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \theta < 1$. (proof by comparing to geometric series)

- $1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$

  $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{x^{n+1}}{x^n}\right| = |x| \leftarrow$ converges absolutely for $|x|<1$.

- $1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots \quad (e^x)$

  $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{n!}{(n+1)!}\left|\frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty}\frac{\cancel{x}}{n+1} = 0 \leftarrow$ converges absolutely on entire real line

- $1 + x + x^2 + x^3 + x^4 + \cdots \quad \left(\frac{1}{1-x}\right)$

  $\lim_{n\to\infty}|x| = |x| \leftarrow$ converges for $|x|<1$

Thm: 37.2 If $f(x)$ is defined by a power series,

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots, \quad I: |x-x_0| < R$$

radius of convergence $\downarrow$ (R)

then

$$f'(x) = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \cdots, \quad I: |x-x_0| < R.$$

Thm: 37.23 If $f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$, $|x-x_0| < R$
$g(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)^2 + \cdots$, $|x-x_0| < R$,

then $f(x) = g(x)$ iff $a_0 = b_0$, $a_1 = b_1$, ...

Thm: 37.24 If $f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$, $|x-x_0| < R$

then $a_0 = f(x_0)$, $a_1 = f'(x_0)$, $a_2 = f''(x_0)$, ...

Taylor series: $f(x) = f(x_0) + f'(x_0)x + \frac{f''(x_0)}{2}x^2 + \frac{f'''(x_0)}{3!}x^3 + \cdots$, $|x-x_0| < R$

Maclaurin series: $x_0 = 0$

Thm: 37.51 Let $y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = Q(x)$.

If each function $f_0(x), f_1(x), \ldots, f_{n-1}(x), Q(x)$ is analytic at $x = x_0$, (i.e. if each function has a Taylor series expansion in powers of $(x-x_0)$ valid for $|x-x_0| < r$), then there is a unique solution $y(x)$ which is also analytic at $x = x_0$, satisfying the n initial conditions

$$y(x_0) = a_0, \quad y'(x_0) = a_1, \ldots, y^{(n-1)}(x_0) = a_{n-1}.$$

Note: polynomials always are analytic everywhere.

Method of successive differentiation:

Use initial conditions and differentiation to keep on solving for the Taylor series coefficients.

Ex. $y'' - (x+1)y' + x^2y = x$, $y(0) = 1$, $y'(0) = 1$.

$$y(x) = y(0) + y'(0) + \frac{y''(0)}{2} + \frac{y'''(0)}{3!} + \cdots \quad \text{(Taylor series)}$$

Let's solve for $y''(0)$ by plugging in above.

$\frac{d}{dx}$

$$y''(0) - 1 = 0 \Rightarrow y''(0) = 1.$$

$\frac{d}{dx}$ $\quad y''(0) - 1 = 0 \Rightarrow y''(0) = 1.$

$$y''' - (x+1)y'' - y' + x^2y' + 2xy = 1$$

$$y'''(0) - 1 - 1 = 1 \Rightarrow y'''(0) = 3$$

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{3x^3}{6} + \cdots$$

Ex. $\quad y''' + \frac{1}{x}y' - \frac{1}{x^2}y = 0, \quad x \neq 0, \quad y(1) = 1, \; y'(1) = 0, \; y''(1) = 1$

Need series solution around $x = 1$.

$$x^{-1} \to -x^{-2} \to 2x^{-3} \to 3!x^{-4} \to -4!x^{-5} \to \cdots$$

$$1 \qquad 1 \qquad 2 \qquad 3! \qquad 4!$$

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-4)^4 - \cdots, \qquad |x-1| < 1$$

$$x^{-2} \to -2x^{-3} \to 6x^{-4} \to -24x^{-5} \to 120x^{-6} \to \cdots$$

$$\frac{1}{x^2} = 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \cdots, \qquad |x-1| < 1$$

So y solved in this power series manner also converges for $|x-1| < 1$.

$$y(x) = y(1) + y'(1)(x-1) + \frac{y''(1)}{2!}(x-1)^2 + \frac{y'''(1)}{3!}(x-1)^3 + \frac{y''''(1)}{4!}(x-1)^4 + \cdots$$

$$y'''(1) = 1$$

$$y'''' + \frac{1}{x}y'' - \frac{1}{x^2}y' - \frac{1}{x^2}y' + \frac{2}{x^3}y'' = 0$$

$$y''''(1) = -1 - 2 = -3$$

$$y(x) = 1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{(x-1)^4}{8} + \cdots$$

Method of undetermined coefficients

Instead of taking derivatives, we just take the Ansatz that y looks like a power series and then solve for the blanks.

Ex. $\quad y'' - (x+1)y' + x^2y = x, \qquad y(0) = 1, \; y'(0) = 1$

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$$

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

$$y''(x) = 2a_2 + 6a_3 x + 12 a_4 x^2 + \cdots$$

$$\begin{array}{lr}
2a_2 + 6a_3 x + 12 a_4 x^2 + \cdots & y'' \\
\quad - a_1 x - 2a_2 x^2 - \cdots & -xy' \\
a_1 - 2a_2 x - 3a_3 x^2 & -y' \\
\quad a_0 x^2 + \cdots & x^2 y \\
\hline
= \quad x &
\end{array}$$

$$\Rightarrow \left.\begin{array}{r} 2a_2 - a_1 = 0 \\ 6a_3 - a_1 - 2a_2 = 1 \\ 12a_4 - 2a_2 - 3a_3 + a_0 = 0 \end{array}\right\} \quad \begin{array}{l} a_2 = \frac{a_1}{2} \\ a_3 = \frac{2a_2 + a_1 + 1}{6} \\ a_4 = \frac{3a_3 + 2a_2 - a_0}{12} \end{array}$$

Recall $a_0 = y(0) = 1$, $a_1 = y'(0) = 1$

$$\Rightarrow \quad a_2 = \frac{1}{2} \quad a_3 = \frac{1}{2}, \quad a_4 = \frac{\frac{3}{2} + 1 - 1}{12} = \frac{1}{8}$$

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{8} + \cdots$$

Ex. $y'' + (\sin x) y' + e^x y = 0$

Ansatz: $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots$$

$$y'' = 2a_2 + 6a_3 x + 12 a_4 x^2 + 20 a_5 x^3 + \cdots$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots$$

$$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \cdots$$

$$\begin{aligned}
&(2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \cdots) \\
&\quad + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots) \\
&\quad + \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots) = 0
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\ &(2a_2 + a_0) + (6a_3 + 2a_1 + a_0)x + \left(12a_4 + 3a_2 + a_1 + \frac{a_0}{2}\right)x^2 \\
&\quad + \left(20a_5 + 4a_3 + a_2 + \frac{a_1}{3} + \frac{a_0}{6}\right)x^3 + \cdots = 0
\end{aligned}$$

$$\Rightarrow\ a_2 = -\frac{a_0}{2}, \quad a_3 = -\frac{2a_1 + a_0}{6} = -\frac{a_1}{3} - \frac{a_0}{6},$$

$$a_4 = -\frac{a_2}{4} - \frac{a_1}{12} - \frac{a_0}{24} = \frac{a_0}{8} - \frac{a_1}{12} - \frac{a_0}{24} = \frac{a_0}{12} - \frac{a_1}{12}$$

$$\begin{aligned}
a_5 &= -\frac{a_3}{5} - \frac{a_2}{20} - \frac{a_1}{60} - \frac{a_0}{120} \\
&= -\frac{1}{5}\left(-\frac{a_1}{3} - \frac{a_0}{6}\right) - \frac{1}{20}\left(-\frac{a_0}{2}\right) - \frac{a_1}{60} - \frac{a_0}{120} \\
&= \frac{a_0}{20} + \frac{a_1}{20}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\ y = &\ a_0\left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{20}x^5 + \cdots\right) \\
&+ a_1\left(x - \frac{1}{3}x^3 - \frac{1}{12}x^4 + \frac{1}{20}x^5 + \cdots\right)
\end{aligned}$$